Rapid inversion of data from 2-D and 3-D resistivity surveys with shifted electrodes

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Outline

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2. Least-squares inversion and Jacobian matrix calculation methods
3. 2-D example
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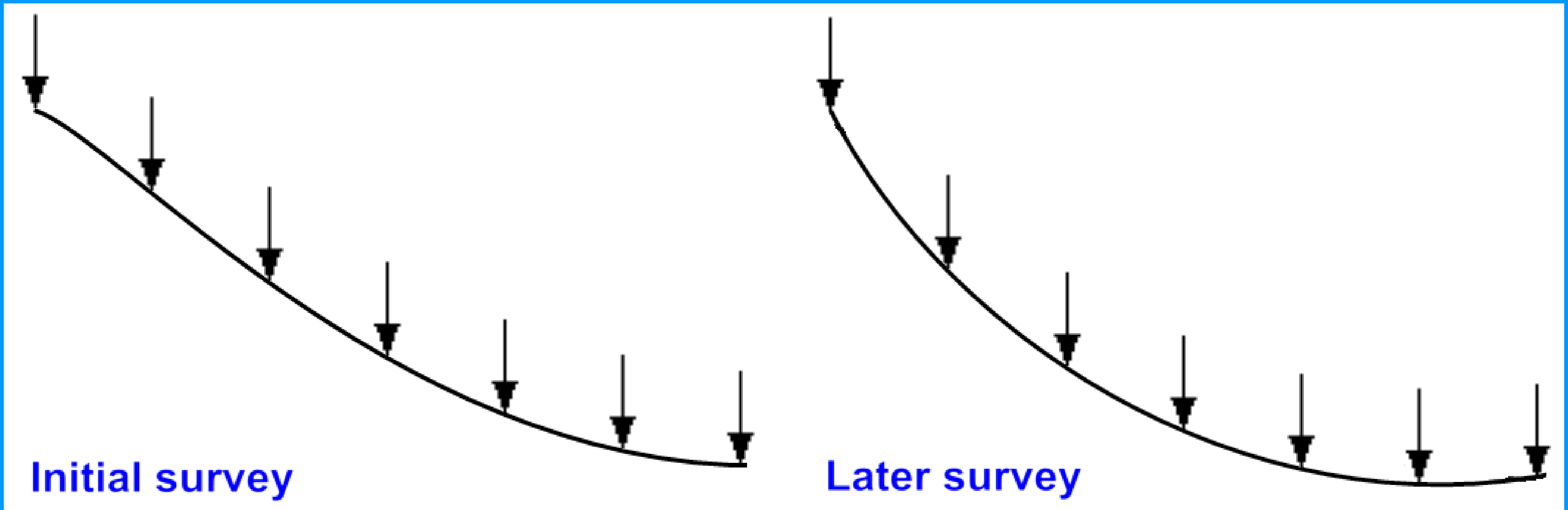
Geoelectrical monitoring surveys

Monitoring surveys are used to detect temporal changes in the subsurface below unstable slopes with repeated measurements. Positions of the electrodes are measured at the start and possibly at regular intervals. Ground movements sometimes occur between the times of the electrode positions measurements. Precise positions of the electrodes are not accurately known and have to be estimated from the resistivity data.



Data available and information needed

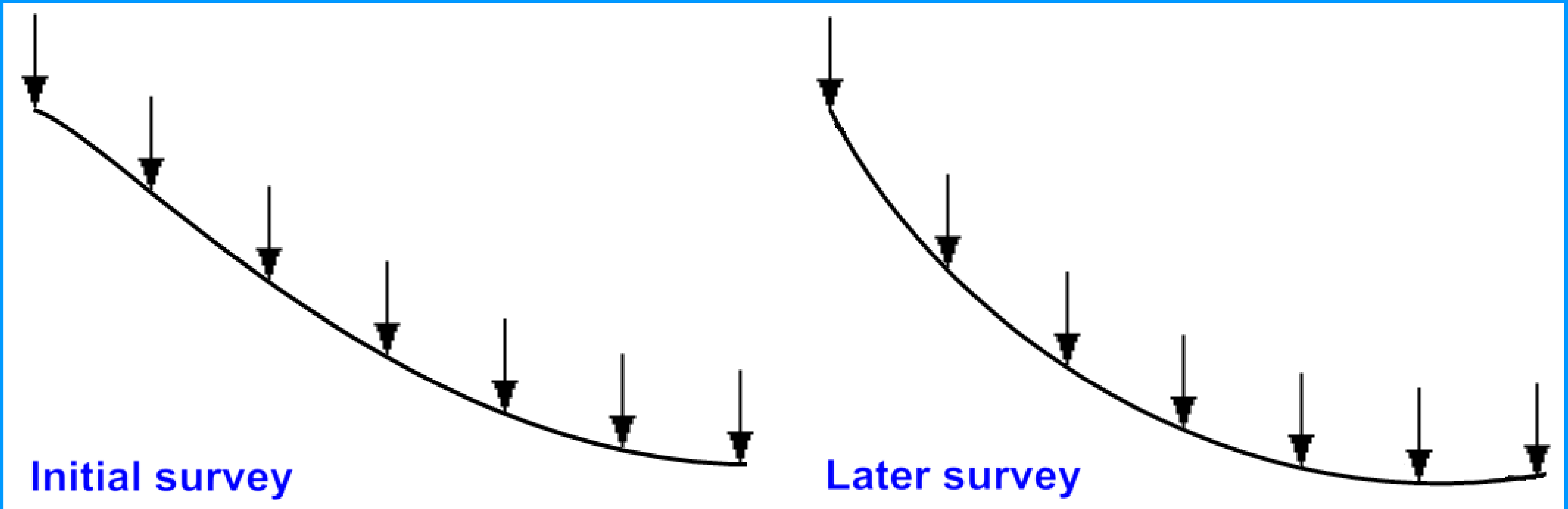
What we have :-

1). Apparent resistivity data and exact position of electrodes from initial survey.

2). Apparent resistivity data from later time survey.

What we want:-

True subsurface resistivity for both surveys, and position of electrodes during later time survey.



The least-squares optimization method

The smoothness-constrained least-squares method is commonly used in 2-D and 3-D resistivity inversion, using the following equation.

$$\left[\mathbf{J}_i^T \mathbf{R}_d \mathbf{J}_i + \lambda_i \mathbf{W}^T \mathbf{R}_m \mathbf{W}\right] \Delta \mathbf{r}_i = \mathbf{J}_i^T \mathbf{R}_d \mathbf{g}_i - \lambda_i \mathbf{W}^T \mathbf{R}_m \mathbf{W} \mathbf{r}_{i-1}$$

W = roughness filter, **λ** = roughness filter damping factor

$\mathbf{r}_{i-1}$ = current inversion model,

$\Delta\mathbf{r}_i$ = change in model resistivity

$\mathbf{R}_m$, $\mathbf{R}_d$ = model and data weighting matrices

g = data misfit, **J** = Jacobian matrix of partial derivatives.

Normally the model parameter vector **r** contains the (logarithm) of the model resistivity values. However, it can be modified to include the position of the electrodes.

Modified least-squares optimization method

The least-squares equation is modified to incorporate the (x,z) positions of the electrodes as model parameters.

$$\left[\mathbf{G}_i^T \mathbf{R}_d \mathbf{G}_i + \lambda_i \mathbf{V}^T \mathbf{R}_m \mathbf{V}\right] \Delta \mathbf{q}_i = \mathbf{G}_i^T \mathbf{R}_d \mathbf{g}_i - \lambda_i \mathbf{V}^T \mathbf{R}_m \mathbf{V}\, \mathbf{q}_{i-1}$$

The model parameters vector **q** becomes

$$\mathbf{q} = \left(r_1 .. r_m \; x_2 .. x_m \; z_2 .. z_m\right) = \left(\mathbf{r} \;\; \mathbf{x} \;\; \mathbf{z}\right)$$

The new Jacobian matrix **G** becomes $\mathbf{G} = \left(\mathbf{J} \;\; \mathbf{X} \;\; \mathbf{Z}\right)$

X and **Z** are the partial derivatives of the (logarithms) of the apparent resistivity values with respect to changes in the (**x,y**) positions of the electrodes.

The new roughness filter term becomes $\mathbf{V} = \left(\mathbf{W} \;\; \alpha \mathbf{W}_x \;\; \beta \mathbf{W}_y\right)$

$\mathbf{W}_x$ and $\mathbf{W}_y$ are the roughness filters for **x** and **z** with relative weights α and β.

Two issues with the modified least-squares method

The modified least-squares equation is

$$\left[\mathbf{G}_i^T\mathbf{R}_d\mathbf{G}_i+\lambda_i\mathbf{V}^T\mathbf{R}_m\mathbf{V}\right]\Delta\mathbf{q}_i=\mathbf{G}_i^T\mathbf{R}_d\mathbf{g}_i-\lambda_i\mathbf{V}^T\mathbf{R}_m\mathbf{V}\mathbf{q}_{i-1}$$

$$\mathbf{G}=\left(\mathbf{J}\ \mathbf{X}\ \mathbf{Z}\right),\quad \mathbf{V}=\left(\mathbf{W}\quad \alpha\mathbf{W}_x\quad \beta\mathbf{W}_y\right)$$

There are 2 issues that needs to be resolved :-

1). Calculation of the spatial Jacobian matrices X and Z.

2). Finding optimum values for the spatial relative damping factors α and β.

Calculating the spatial Jacobian matrices

The terms in the X spatial Jacobian matrix contains the change in the potential *Φ* due to a change the *x* position of the electrode, such as

$$\frac{\partial \phi_i}{\partial x_2}$$

There are 2 methods that are commonly used to calculate the sensitivity values, the 'perturbation' and 'adjoint-equation' methods. The potentials values are calculated using the finite-element method by solving the matrix equation : - CΦ = s .

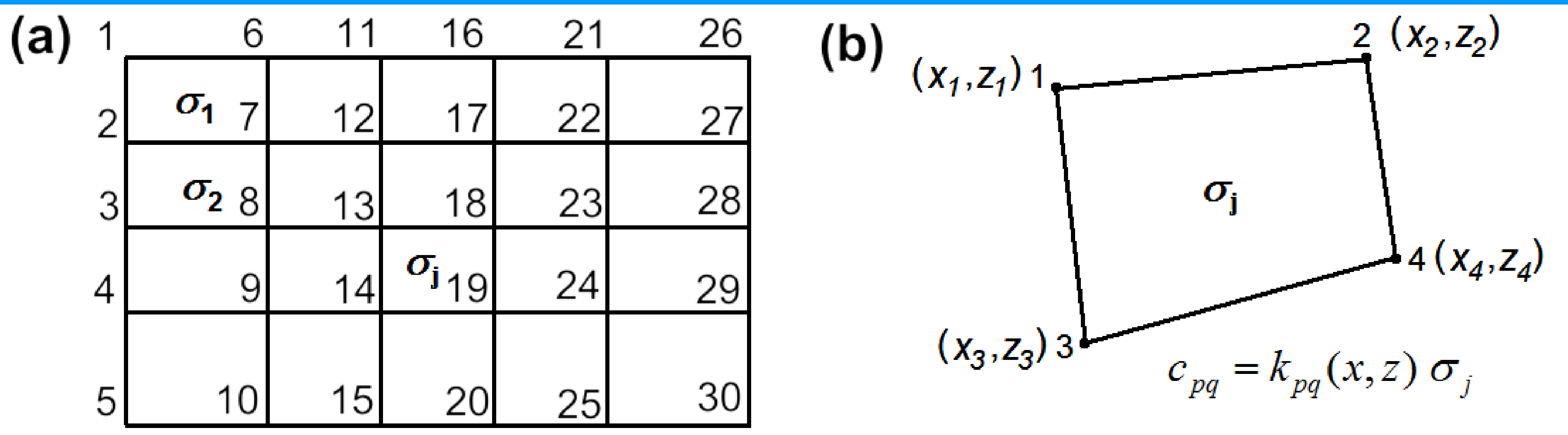
C = capacitance matrix that contains resistivity distribution and mesh shapes

Φ = vector with the potentials at the nodes

s = vector with current at the nodes.

A closer look at the finite-element method

The subsurface is divided into a large number of cells. The potentials at the nodes are calculated by solving the capacitance matrix equation $\mathbf{C\Phi=s}$. The $\mathbf{C}$ matrix contains the coupling coefficients between the nodes in the elements of the mesh. For a single mesh element, they have the form $c_{pq} = k_{pq}(x,z)\sigma_j$, where σ_j is cell conductivity. The $k_{pq}(x,z)$ term only depends on the coordinates at the corners of the cell element.



Perturbation method – a brute force approach

The perturbation method calculates partial derivatives by re-solving the finite-element matrix equation **CΦ = s** with a small change in the electrode position, such as

$$\frac{\partial \phi_i}{\partial x_2} \approx \frac{\phi_i\left(x_2 + \Delta x_2\right) - \phi_i\left(x_2\right)}{\Delta x_2}$$

The potentials only depend on the relative positions of the electrodes. We can fix the first electrode position. For a survey with 101 electrodes, this method recalculates the potentials 200 times for 2-D problems (300 times for 3-D).

Advantage : Simple to implement in computer program.
Disadvantage : Very slow, impractical for 3-D. Possible directional bias. Not used for calculating resistivity Jacobian.

Adjoint-equation method - theory

A more efficient method is the adjoint-equation method. Differentiation of the capacitance matrix equation $\mathbf{C\Phi} = \mathbf{s}$ wrt the x-position of an electrode gives

$$\mathbf{C}\frac{\partial \mathbf{\Phi}}{\partial x_k} = -\frac{\partial \mathbf{C}}{\partial x_k}\mathbf{\Phi}$$

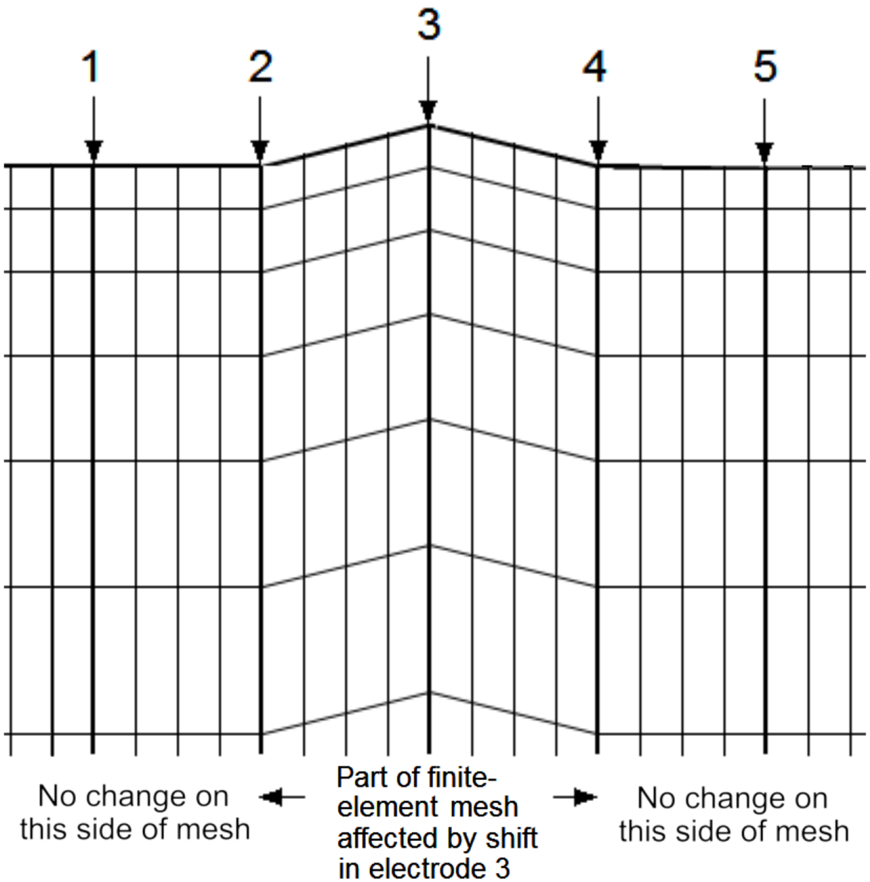
It has the same form as the capacitance matrix equation. All the information needed to calculate $\partial\mathbf{\Phi}/\partial x_k$ is available in the process of solving $\mathbf{C\Phi} = \mathbf{s}$ to calculate the potentials $\mathbf{\Phi}$. It is not necessary to directly re-solve the matrix equation. $\partial\mathrm{C}/\partial x_k$ can be calculated from $c_{pq} = k_{pq}(x,z)\sigma_j$ such as

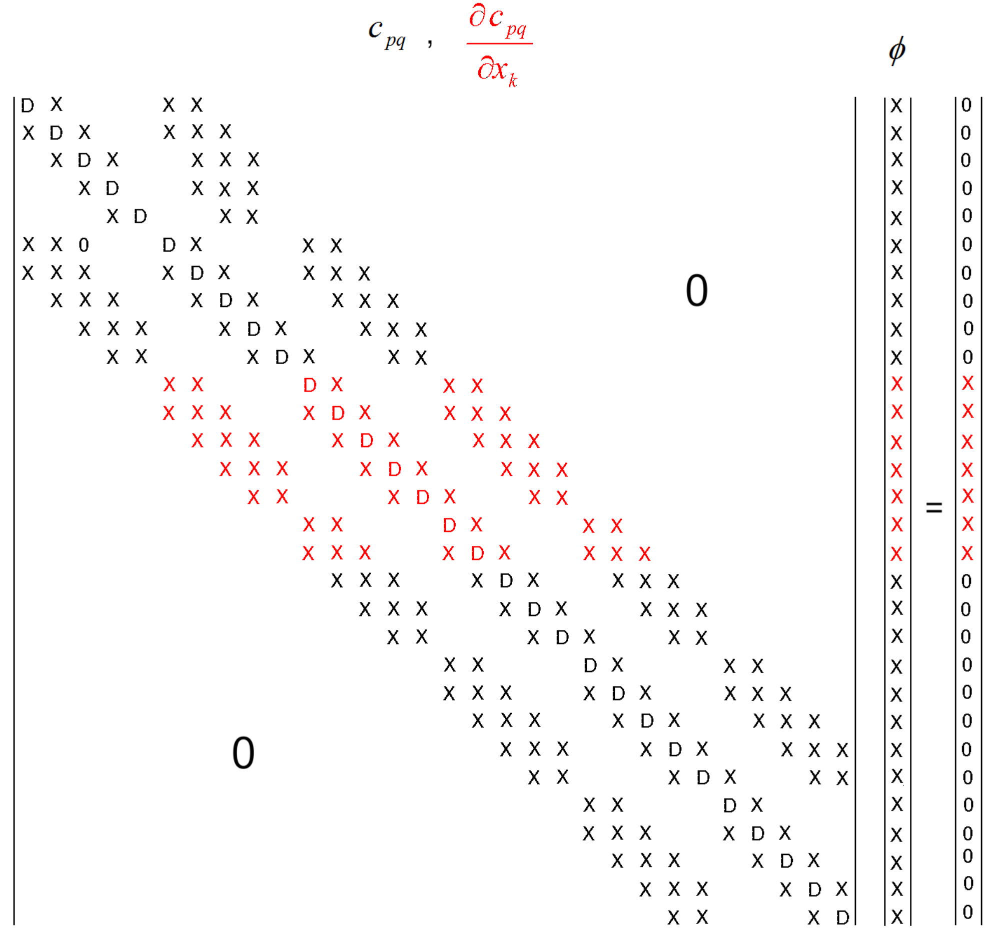
$$\frac{\partial c_{pq}}{\partial x_k} \approx \frac{k_{pq}(x_k + \Delta x_k, z) - k_{pq}(x_k - \Delta x_k, z)}{2\Delta x_k}\sigma_j$$

A two-sided difference is used to avoid directional bias.

Adjoint-equation method - efficiency

Each row in the C matrix has 9 (or less) non-zero elements for a 2-D mesh. Only a small number of rows in $\partial C/\partial x_k$ matrix are affected by a shift in an electrode. The non-zero row are multiplied by the Φ vector to obtain $\partial \Phi / \partial x_k$.





Adjoint-equation method - comparison

A 2-D mesh with n_x and n_z nodes in the x and z directions, and e electrodes, m data points, and *4* nodes between adjacent electrodes. Number of operation required :-

Adjoint equation ≈ $90.m.\ n_z$

Perturbation ≈ $0.5.\ n_x.(n_z+1).(n_z+4) + 4.(e-1).\ n_z.\ n_x.(n_z+2)$

Example :- Survey line with *50* electrodes and *469* data points, mesh with *237* by *34* nodes.

Adjoint equation method ≈ $1.41x10^8$ operations

Perturbation method ≈ $6.21x10^9$ operations

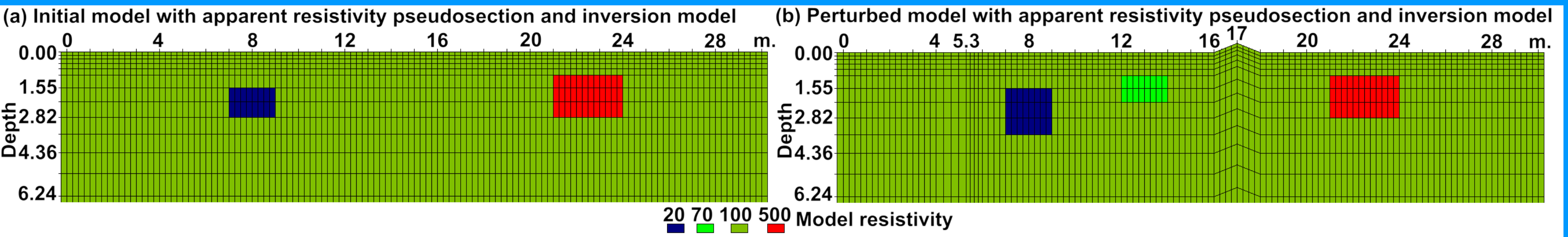
Adjoint equation method is about 44 times faster.

Speed advantage of adjoint equation method increases with mesh size and number of electrodes, particularly for 3-D problems.

Synthetic 2-D model test

The initial model has a survey line with 31 electrodes with a uniform 1 m spacing on a flat surface, a shallow 500 ohm.m rectangular block and a deeper 20 ohm.m block in a 100 ohm.m medium.

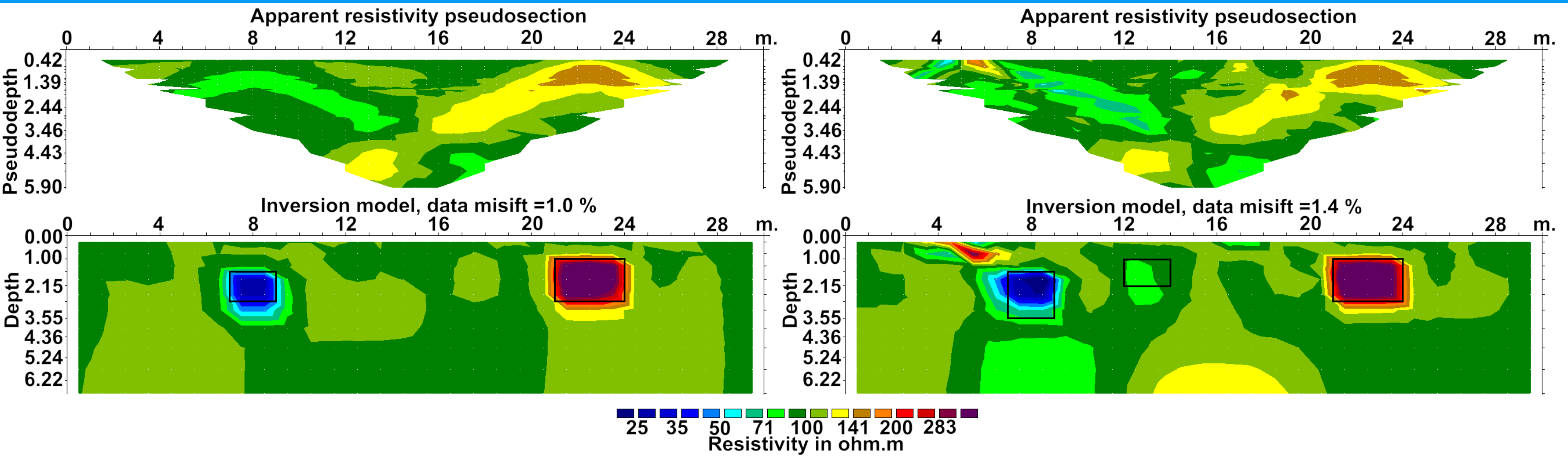
The perturbed model has 4 changes. The electrode at the 5.0 m is shifted 0.3 m to the right, and the electrode at the 17.0 m mark is shifted 0.4 m. upwards. A 70 ohm.m prism is added between the two existing prisms. The 20 ohm.m low resistivity prism is extended downwards by 0.7 m.



Inversion model with fixed electrodes

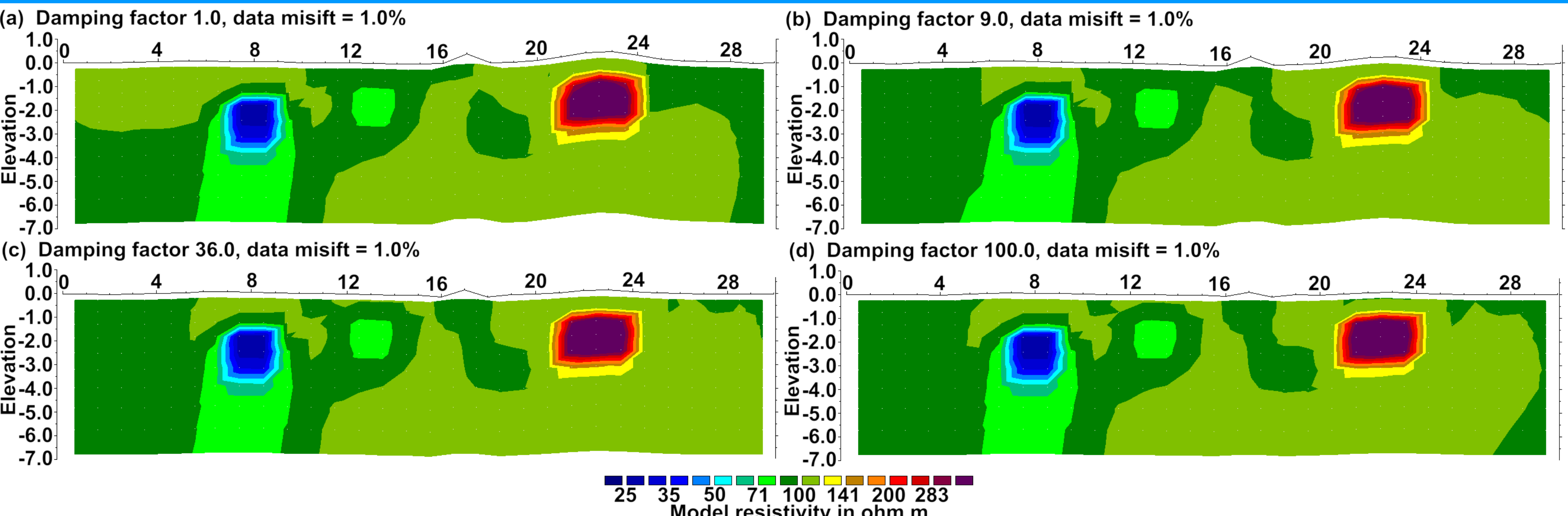
The data set consists of dipole-dipole arrays with the '*a*' dipole lengths ranging from 1 to 4 m., and the '*n*' factor ranging from 1 to 6. Voltage dependent random added to the data, which resulted in average noise level of 1.0% for the apparent resistivity data set. There is a good fit for initial model data set, but the model for the perturbed data set has significant distortions.

Original model   Perturbed model



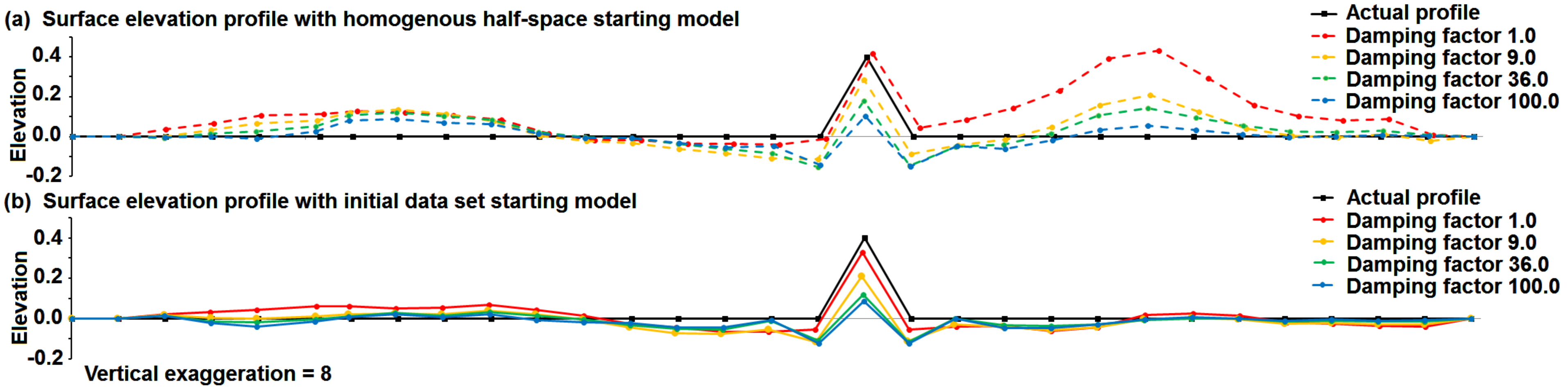
Inversion model with variable electrodes

The electrodes are allowed to move. A homogenous half-space initial model is used. Tests were carried out using values of 1 to 100 for the relative spatial damping factors (α,β). For small values of α and β, there is a significant topographic distortion over the shallow high-resistivity block. It is reduced with higher spatial damping factors, but with reduced accuracy in the electrodes positions.



Closer look at the surface profile

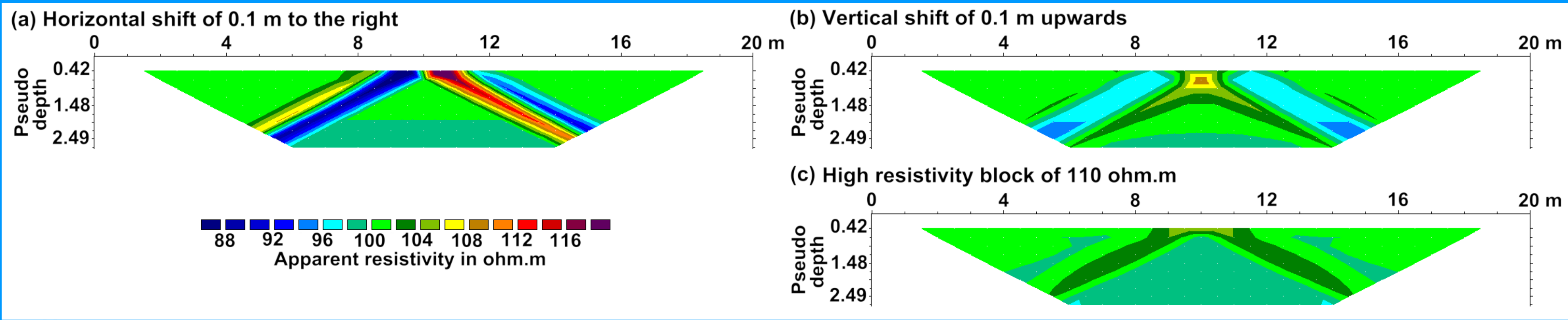
The distortion in the surface profile over the high resistivity prism is clearly shown in the profile plot (a). The inversion algorithm is not able to fully distinguish between the effects of vertical shifts in the electrodes and a shallow subsurface high resistivity anomaly. The distortion is reduced when the damping factor is increased. However, increasing the higher damping factor also decreases the elevation at the electrode located at the 17 m. mark compared to the true elevation (black line).



Reason for topographic distortion

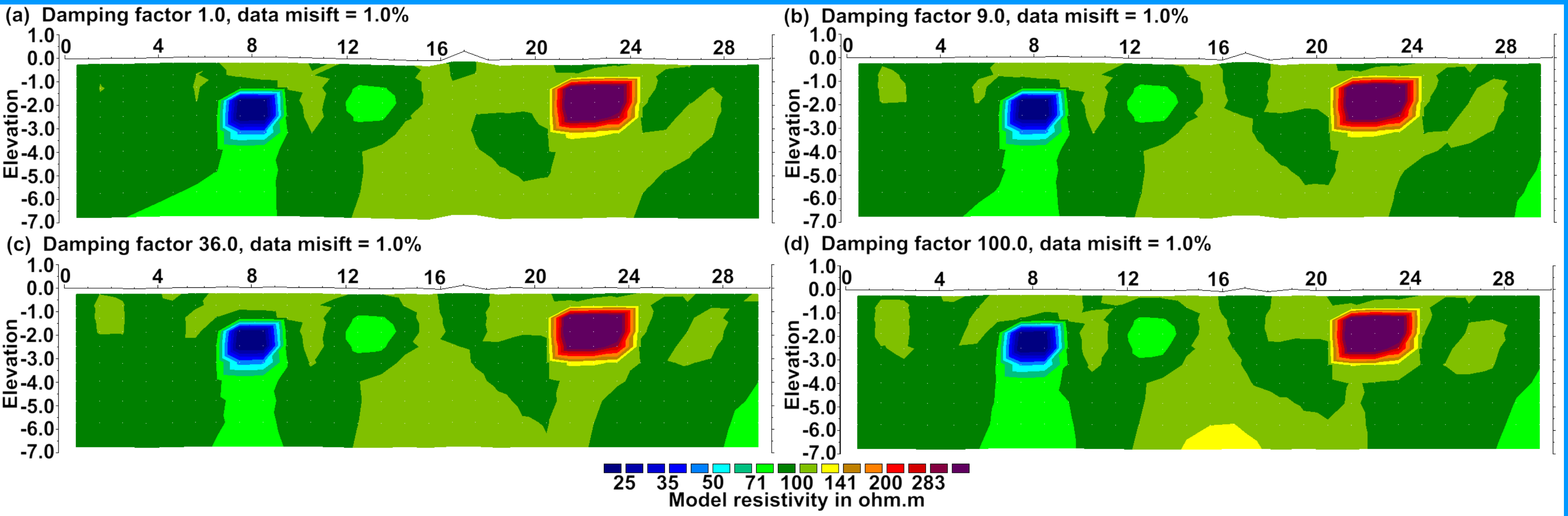
Below are apparent resistivity pseudosections for a homogenous half-space with (a) horizontal and (b) vertical electrode shifts, (c) a shallow high resistivity block.

A horizontal shift of an electrode shows a distinctive pattern with a low anomaly next to a high anomaly. This can be easily distinguished from subsurface resistivity variations by the inversion program. An upwards electrode shift has an anomaly pattern similar to a shallow high resistivity block, thus it is difficult to distinguish them.



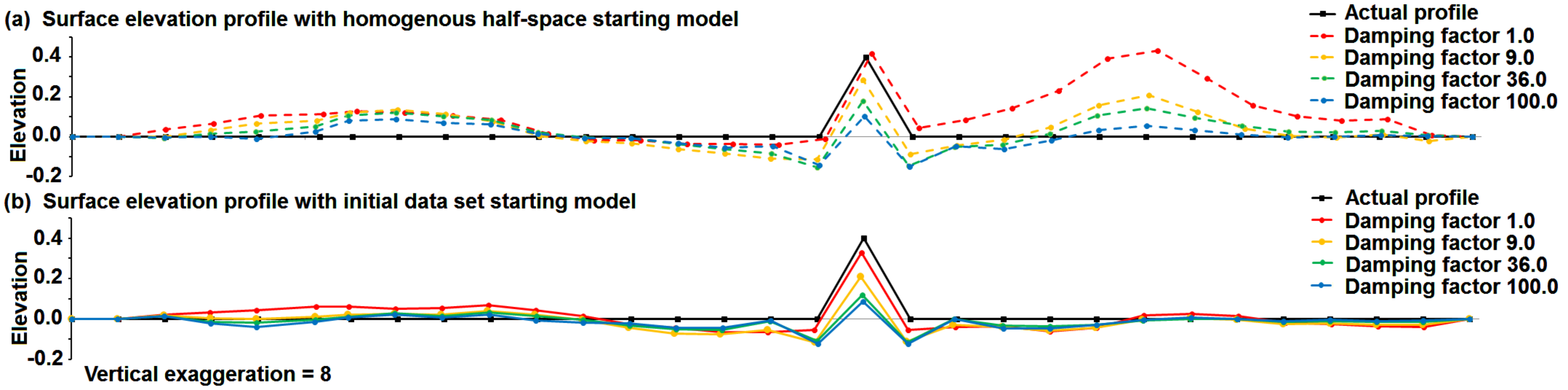
Method to reduce topographic distortions

The problem of shifting electrodes frequently occur in a time-lapse survey. Normally the positions of the electrode are accurately measured at the beginning. Instead of using a homogenous half-space as the starting model, we use the model from the first time-lapse data set inversion. The distortion over the high resistivity block is eliminated, even with the lowest relative spatial damping factor of 1.0.



Another look at the surface profile

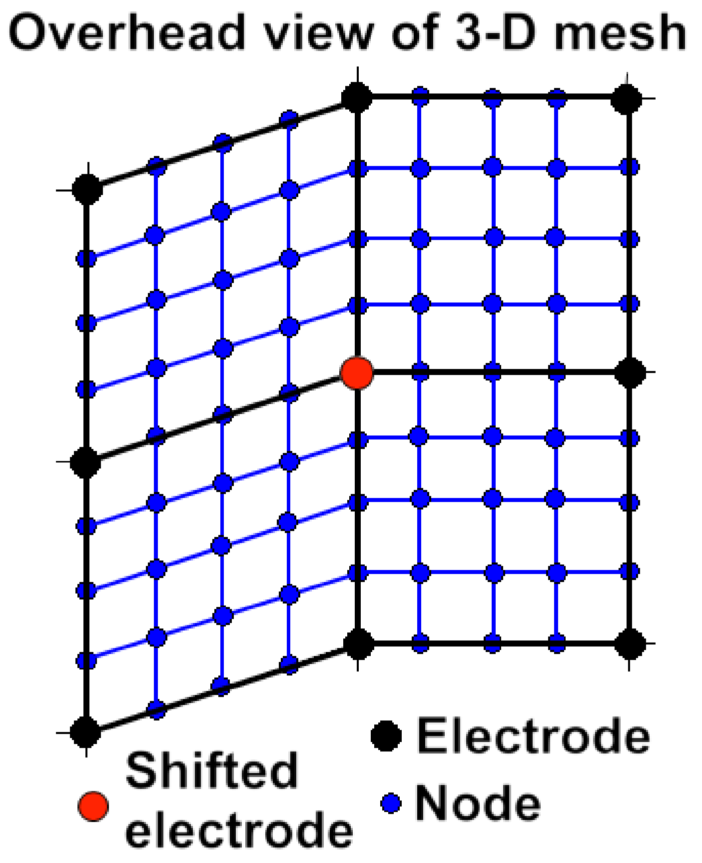
The profile is essentially flat over the high resistivity block in the modified inversion method (b). The most accurate results are obtained using a relative spatial damping factor of 1.0. The modified inversion method basically uses the difference between the apparent resistivity of the later and initial data sets. Temporal changes in the subsurface are usually much smaller than spatial variations. This method makes it easier to distinguish electrode shifts from resistivity variations.



3-D surveys

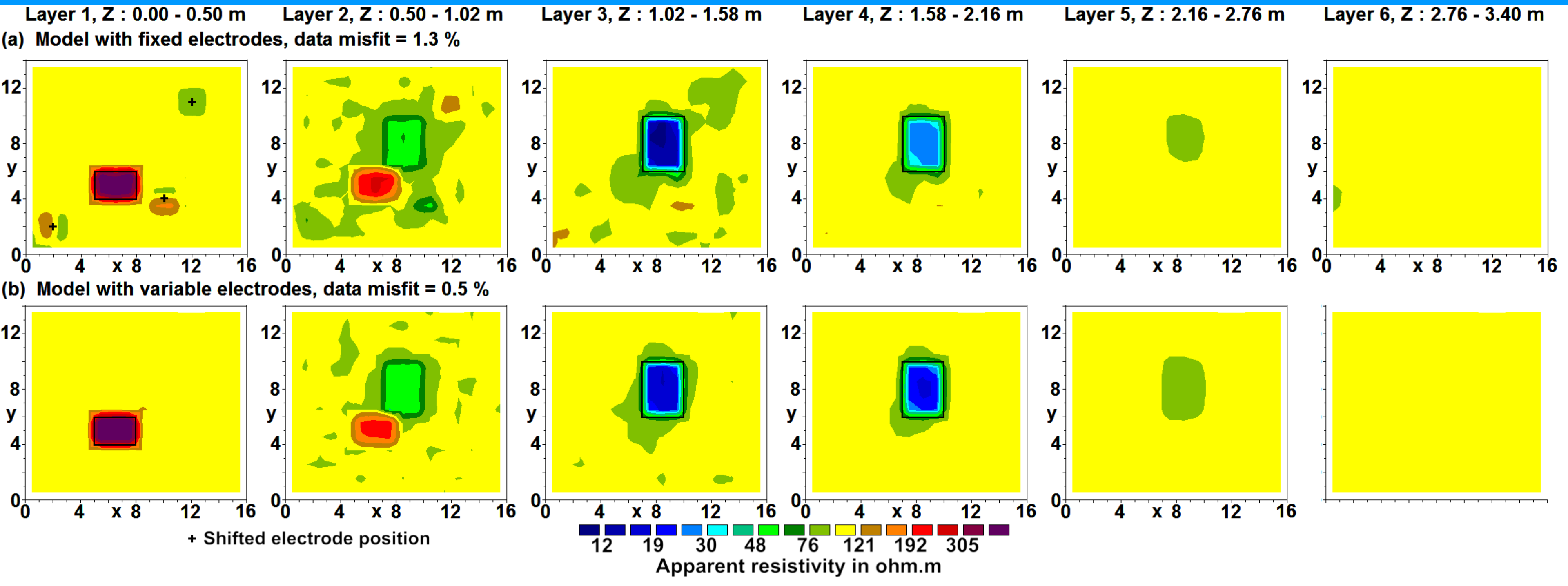
The same techniques can be used for 3-D surveys. A shifted electrode will affect the finite-element cells located between the electrode and 8 surrounding electrodes. If 4 nodes are used between adjacent electrodes, the number of mesh cells affected is $64n_z$, compared to $8n_z$ for 2-D.

A similar method can be used to calculate the Jacobian matrix elements by calculating the change in the capacitance matrix for the nodes affected. As the calculation time for 3-D models can be 100 times slower than 2-D models, the use of the adjoint-equation method is crucial.



3-D example

The synthetic model has 2 blocks of 20 and 400 ohm.m in a 100 ohm.m medium with a survey grid using 17 by 15 electrodes at 1 m apart. Two electrodes were shifted 0.3 m horizontally and one electrode vertically upwards by 0.4 m. The inversion model with fixed electrodes (a) shows artifacts at the positions of the shifted electrodes which are removed in the model with variable electrodes (b).



Conclusions

(a) The Jacobian matrices for horizontal and vertical shifts in the electrode positions can be rapidly calculated using the adjoint equation method. This is essential in 3-D surveys where shifts in 3 directions occur for a larger number of electrodes compared to 2-D surveys, and the forward modelling routine can be 100 times slower.

(b) For time-lapse surveys, the accuracy of the recovered electrode positions can be greatly improved by using the inversion model from an initial data set as the starting model for the inversion of a later time data set. The initial data set has accurately measured electrode positions and provides a good starting model for the subsurface resistivity.